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Energy and momentum density of a Landau-damped wave packet

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The energy of a Landau-damped electrostatic wave is a long standing problem. Calculations based on a spatially infinite wave are deficient. In this paper, a wave packet is analysed. The energy density depends on second-order initial conditions that are independent of the first-order wave. Pictures of energy and momentum transfer to resonant electrons are presented. On physical grounds, suitable second-order initial conditions are proposed. The resulting wave energy agrees with fluid theory. The ratio between energy and momentum is not the phase velocity u_p as predicted by fluid theory, but $u_p/2$ in agreement with Landau-damping physics.

1. Introduction

Wave energy and momentum expressions for a collisionless plasma have been derived from Maxwell's equations in which the plasma properties are described by the dielectric permittivity (Bers 1975). The plasma is supposed to be a linear medium, i.e. terms bilinear in the field are retained, for example Ej, in which the current j is proportional to the electric field E, but second-order terms due to non-linear interaction are not taken into account. But the response of a collisionless plasma is not linear, and is not fully described by the dielectric permittivity. Nevertheless, in textbooks (Davidson 1972, Swanson 1989) these dielectric fluid expressions are presented as valid for a Vlasov plasma. The aim of this paper is to derive energy and momentum transfer in Landau-damped waves from the Vlasov-Poisson equations, expanded to second-order in the field amplitude.

Lacina (1972) has considered an infinite damped electrostatic travelling wave, and calculated the associated kinetic energy density U_s and momentum density P_s of particle species s from the equations of motion, which are equivalent to the Vlasov equation. His expressions for U_s and P_s (see Appendix A) are independent of the second-order field. The wave momentum density $\sum P_s$ and energy density $U_f + \sum U_s$, in which U_f is the field energy density, are constant if the dispersion relation is satisfied. The values of these constants, however, have little physical significance. Some terms in the expressions can be associated with non-resonant and resonant motion. The sum of the energy densities of the field and the damped non-resonant motion amounts to $U_f \partial_\omega \omega \epsilon_r$, in which ϵ_r is the real part of the dielectric function, in agreement with fluid theory. Also, the momentum density of the damped non-resonant motion $U_f k \partial_\omega \epsilon_r$ agrees. Though some terms can be attributed physical significance, the expressions leave energy and momentum transfer and the role of initial conditions for further investigation.

In this paper, the Landau wave is assumed to be nearly periodic in space, with zero average in first order, so its k spectrum has only non-zero components. Then, to express the electronic part of the energy density U_e in terms of the distribution function governed by Vlasov's equation, one has to expand to second order, because first-order terms drop out when averaged over a wavelength. $U_e = N_e \int \frac{1}{2} m v^2 \bar{f}_{e2} \mathrm{d}v$ in which N_e is the electron density and $\bar{f}_{e2}$ is the slowly varying part of the second-order electron distribution function, due to non-linear interaction of the fast varying (k, ω) components with their complex conjugates $(-k, -\omega)$. Best (1985) showed that for a wave packet (continuous peaked k spectrum), $\bar{f}_{e2}$ contains a term proportional to $\bar{E}_2$, the slowly varying part of the second-order field. In a sequel (Best 1999), a problem with secular terms was solved, and other theories (Morrison and Pfirsch 1992, Krasovskiĭ 1995) on wave energy were discussed. It was found that a spatially infinite wave (discrete k spectrum) is a singular limit of a wave packet. In this limit $\bar{E}_2$ can be zero, but even then the term in U_e proportional to $\bar{E}_2$ does not vanish, because another factor tends to infinity. The energy density tends to a constant determined by the initial value of the second-order distribution function; it is undetermined if only first-order quantities are taken into account.

The problem to calculate U_e in Landau theory is ill posed in a sense, since it depends on initial conditions for $\bar{f}_{e2}$ that are independent of the first-order wave parameters. However, fluid theory, in which U_e is well defined, cannot be derived from the fundamental Vlasov equations without making extra physical assumptions. The problem in Vlasov theory is really to find suitable second-order initial conditions on physical grounds. For this, a wave-packet model is more suitable than an infinite wave.

Ion dynamics are included because ion momentum exchange with electrons cannot be ignored, while for the energy balance the heavy ions can be treated as a fixed neutralizing background. The derivation is self-contained, i.e. previous papers are referred to but results are rederived.

1.1. Basic equations

The basic equations are the Vlasov-Poisson system

$$(\partial_t + v\partial_x)f_s + \frac{e_s}{m_s}E\partial_v f_s = 0, \quad \partial_x E = \frac{Ne}{\epsilon_o}\int (f_e - f_i)\mathrm{d}v \tag{1.1}$$

for the species s (electrons e and ions i). The electron charge is $e = e_e = -e_i$. In equilibrium the density N of both species is constant, and their normalized distributions are Maxwellian:

$$F_s = \Big(\frac{m_s}{2\pi\theta}\Big)^{1/2}\mathrm{e}^{-m_s v^2/2\theta}, \quad \int F_s\mathrm{d}v = 1.$$

The distributions and the field are expanded to second order as

$$f_s = F_s + \int f_{s1}\mathrm{e}^{\mathrm{i}(kx-\omega t)}\mathrm{d}k\mathrm{d}\omega + \int f_{s0}\mathrm{e}^{\mathrm{i}(k''x-\omega'' t)}\mathrm{d}k\mathrm{d}k'\mathrm{d}\omega\mathrm{d}\omega',$$

$$E = \int E_1\mathrm{e}^{\mathrm{i}(kx-\omega t)}\mathrm{d}k\mathrm{d}\omega + \int E_0\mathrm{e}^{\mathrm{i}(k''x-\omega'' t)}\mathrm{d}k\mathrm{d}k'\mathrm{d}\omega\mathrm{d}\omega' \tag{1.2}$$

with first-order $f_{s1}(k,v,\omega)$ and $E_1(k,\omega)$ and second-order $f_{s0}(k,k',v,\omega,\omega')$ and $E_0(k,k',\omega,\omega')$; $k'' = k + k'$ and $\omega'' = \omega + \omega'$. The integration region of k' and ω' is confined to $k' \approx -k$ and $\omega' \approx -\omega$, i.e. small k'' and ω'', for which f_{s0} and E_0 differ from zero. Other second-order terms (second harmonics) are not considered. The wave energy and momentum densities read

$$U = \frac{N}{2}\int v^2(m_e f_{e0} + m_i f_{i0})\mathrm{e}^{\mathrm{i}(k''x-\omega'' t)}\mathrm{d}v\mathrm{d}k\mathrm{d}k'\mathrm{d}\omega\mathrm{d}\omega' + U_f,$$

$$P = N\int v(m_e f_{e0} + m_i f_{i0})\mathrm{e}^{\mathrm{i}(k''x-\omega'' t)}\mathrm{d}v\mathrm{d}k\mathrm{d}k'\mathrm{d}\omega\mathrm{d}\omega'. \tag{1.3}$$

$U_f(x,t)$ is the field energy density

$$U_f = \frac{\epsilon_o}{2}\int_0 E_1 E_1'\mathrm{e}^{\mathrm{i}(k''x-\omega'' t)}\mathrm{d}k\mathrm{d}k'\mathrm{d}\omega\mathrm{d}\omega' \tag{1.4}$$

in which $E_1' = E_1(k',\omega')$. The 0 attached to the integral indicates that the integration region is confined to small k'' and ω'' to average over a wavelength. E_1 is supposed to represent a damped electron wave packet with frequency near the plasma frequency ω_p:

$$E_1 = \frac{\mathrm{i}}{2\pi}\frac{E_k}{\omega - \omega_0}, \quad \omega_0 = \omega_r - \mathrm{i}\gamma. \tag{1.5}$$

The k-spectrum E_k consists of two peaks at $k = \pm k_0$ with $\omega_p^2/k_0^2 >> \theta/m_e$. The Landau pole ω_0 is discussed in section 3.1. The frequency $\omega_r \approx \omega_p \mathrm{sgn}k$, which makes the phase velocity $u_p = \omega_r/k$ positive, and γ is the small damping decrement ($0 < \gamma << \omega_p$). Substitution of (1.5) in (1.4), integration and use of the approximation $\omega_r' = -\omega_r + k''\partial_k\omega_r$ gives

$$U_f = \frac{\epsilon_o}{2}\int_0 E_k E_{k'}\mathrm{e}^{\mathrm{i}k''(x-u_g t)-2\gamma t}\mathrm{d}k\mathrm{d}k'' = \int U_{k''}\mathrm{e}^{\mathrm{i}k''(x-u_g t)-2\gamma t}\mathrm{d}k'', \tag{1.6}$$

in which $u_g = \partial_k\omega_r$ is the group velocity and $U_{k''} = (\epsilon_o/2)\int_0 E_k E_{k''-k}\mathrm{d}k$ is the spectrum of $U_0 = U_f(x,0)$.

2. Normal modes

The distributions are written as (Best 1974)

$$f_{s1} = \frac{\mathrm{i}\epsilon_o}{Ne_s} E_1 K_s, \quad f_{s0} = \frac{\mathrm{i}\epsilon_o}{Ne_s} E_0 K_s'' - \frac{\epsilon_o}{2Nm_s} E_1 E_1' L_s, \tag{2.1}$$

with $K_s(k, v, \omega)$, $K_s'' = K_s(k'', v, \omega'')$ and $L_s(k, k', v, \omega, \omega')$. Substitution of (1.2) and (2.1) in (1.1) gives, collecting first- and second-order terms respectively, equations for K_s and L_s which replace the Vlasov-Poisson equations for f_{s1} and f_{s0}:

$$(kv - \omega)K_s = \omega_s^2 F_s', \quad \int (K_e + K_i)\mathrm{d}v = k,$$

$$(k''v - \omega'')L_s = \partial_v(K_s + K_s'), \quad \int (L_e + L_i)\mathrm{d}v = 0 \tag{2.2}$$

in which $\omega_s^2 = Ne_s^2/\epsilon_o m_s$. Explicit expressions for the Van Kampen (1955) normal modes K_s, which are singular at $kv - \omega = 0$, are given in (3.6). Only a few momenta of K_s and L_s are needed to compute P and U. Manipulation of (2.2) gives momenta summed over species:

$$\int v(K_e + K_i)\mathrm{d}v = \omega, \quad \int v^2(K_e + K_i)\mathrm{d}v = \frac{\omega^2 - \omega_p^2}{k},$$

$$\int v(L_e + L_i)\mathrm{d}v = 0, \quad \int v^2(L_e + L_i)\mathrm{d}v = -1, \quad k'' \neq 0 \tag{2.3}$$

in which $\omega_p^2 = \omega_e^2 + \omega_i^2$. Substitution from (2.1), (2.3) and (1.4) in (1.3) gives

$$U = \frac{\mathrm{i}\epsilon_o}{2e} \int E_0 v^2 (m_e K_e'' - m_i K_i'') \mathrm{e}^{\mathrm{i}(k''x - \omega''t)} \mathrm{d}v\mathrm{d}k\mathrm{d}k'\mathrm{d}\omega\mathrm{d}\omega' + \frac{3}{2}U_f,$$

$$P = \frac{\mathrm{i}\epsilon_o}{e} \int E_0 v (m_e K_e'' - m_i K_i'') \mathrm{e}^{\mathrm{i}(k''x - \omega''t)} \mathrm{d}v\mathrm{d}k\mathrm{d}k'\mathrm{d}\omega\mathrm{d}\omega'. \tag{2.4}$$

The momenta of K_s depend on initial conditions for a multi-species plasma (Best and Lambert 1981). But this dependence is negligible if $\omega^2/k^2 >> \theta/m_i$ which certainly applies to the electron wave moving at the phase velocity $u_p = \omega_{r0}/k_0$ with $\omega_{r0} = \omega_r(k_0)$. Then the singularity in $K_i = \omega_i^2 F_i'/(kv - \omega)$ can be ignored. One has

$$\int K_i \mathrm{d}v = -k\chi_{ir}, \quad \int K_e \mathrm{d}v = k(1 + \chi_{ir}) \tag{2.5}$$

in which χ_{ir} is the ion susceptibility. A general expression (3.5) for $\int K_s \mathrm{d}v$ is derived in the next section. Equations (2.5) proves to be valid also if $\omega^2/k^2 = \mathcal{O}(\theta/m_i)$, provided that the initial ion disturbance is minimal. Higher momenta are determined by

$$\int vK_s \mathrm{d}v = \frac{\omega}{k} \int K_s \mathrm{d}v, \quad \int v^2 K_s \mathrm{d}v = \frac{\omega^2}{k^2} \int K_s \mathrm{d}v - \frac{\omega_s^2}{k}. \tag{2.6}$$

Substitution of (2.6) and (2.5) in (2.4) yields

$$U = \frac{\mathrm{i}m_i\epsilon_o}{2e} \int_0 \frac{\omega''^2}{k''} E_0 \chi_{ir}(k'', \omega'') \mathrm{e}^{\mathrm{i}(k''x - \omega''t)} dk\mathrm{d}k'\mathrm{d}\omega\mathrm{d}\omega' + \frac{3}{2}U_f,$$

$$P = \frac{\mathrm{i}m_i\epsilon_o}{e} \int_0 \omega'' E_0 \chi_{ir}(k'', \omega'') \mathrm{e}^{\mathrm{i}(k''x - \omega''t)} dk\mathrm{d}k'\mathrm{d}\omega\mathrm{d}\omega' \tag{2.7}$$

to leading order in m_e/m_i. In the approximation $\chi_{ir} = -\omega_i^2/\omega^2$, which will be justified in section 4 for U, $m_i\chi_{ir}(k'', \omega'')$ can be replaced by $-Ne^2/\epsilon_o\omega''^2$, in agreement with previous results (Best 1999).

3. Initial-value problem

At $t = 0$, the distributions are assumed to have the form

$$f_s(x, v, 0) = F_s + \int g_{s1} \mathrm{e}^{\mathrm{i}kx} \mathrm{d}k + \int g_{s0} \mathrm{e}^{\mathrm{i}k'' x} \mathrm{d}k \mathrm{d}k' \tag{3.1}$$

with $g_{s1}(k, v)$ and $g_{s0}(k, k', v)$. Putting $t = 0$ in (1.2) and substituting (2.1) gives integral equations for E_1 and E_0:

$$\mathrm{i} \int E_1 K_s \mathrm{d}\omega = \frac{N e_s}{\epsilon_o} g_{s1}, \quad \mathrm{i} \int E_0 K_s'' \mathrm{d}\omega \mathrm{d}\omega' = \frac{e_s}{2 m_s} \int_0 E_1 E_1' L_s \mathrm{d}\omega \mathrm{d}\omega' + \frac{N e_s}{\epsilon_o} g_{s0}. \tag{3.2}$$

3.1. First order

E_1 is calculated conveniently by aid of the relation

$$\int \frac{K_s}{\Omega_+ - kv} \mathrm{d}v = \frac{1}{\Omega_+ - \omega} \int K_s \Big(1 + \frac{kv - \omega}{\Omega_+ - kv}\Big) \mathrm{d}v$$

$$= \frac{1}{\Omega_+ - \omega} \int \Big(K_s + \frac{\omega_s^2 F_s'}{\Omega_+ - kv}\Big) \mathrm{d}v = \frac{\int K_s \mathrm{d}v + k \chi_s^+(k, \Omega)}{\Omega_+ - \omega}, \tag{3.3}$$

in which (2.2) has been used and $\Omega_+ = \Omega + \mathrm{i}0$ indicates the integration path around the pole. The $\chi_s^+(k, \omega) = (\omega_s^2/k) \int F_s' \mathrm{d}v/(\omega_+ - kv)$ are susceptibilities.

Division of the first equation (3.2) by $2\pi(\Omega_+ - kv)$, integration over v and summation over s yields the 'positive' part E_1^+, defined as $E_1^+(k, \omega) = (\mathrm{i}/2\pi) \int E_1(k, w) \mathrm{d}w/(\omega_+ - w)$; thus

$$\frac{\mathrm{i}}{2\pi} \int E_1 \frac{K_e + K_i}{\Omega_+ - kv} \mathrm{d}v \mathrm{d}\omega = k \epsilon^+(k, \Omega) E_1^+(k, \Omega) = \frac{N e}{2\pi \epsilon_o} \int \frac{g_{e1} - g_{i1}}{\Omega_+ - kv} \mathrm{d}v, \tag{3.4}$$

in which the permittivity $\epsilon^+(k, \omega) = 1 + \sum \chi_s^+(k, \omega)$.

E_1^+ determines the first-order field for $t > 0$, as can be seen from

$$\int \frac{E_1}{\Omega_+ - \omega} \mathrm{d}\omega = -\mathrm{i} \int_0^\infty \mathrm{d}t \mathrm{e}^{\mathrm{i}\Omega t} \int \mathrm{d}\omega E_1 \mathrm{e}^{-\mathrm{i}\omega t}.$$

The asymptotic behaviour is determined by the zero point ω_0 near the real axis of the analytic continuation of $\epsilon^+(k, \Omega)$ into the lower half of the complex ω plane, provided that the g_{s1} are smooth functions of v. This leads to (1.5) with well known expressions $\omega_r^2 \approx \omega_p^2 + 3k^2\theta/m_e$ and $\gamma \approx -(\pi \omega_p^3/2k^2) F'(\omega_r/k)$.

K_s can be written in the form

$$K_s = \omega_s^2 F_s' \frac{\mathcal{P}}{kv - \omega} + \Big(\int K_s \mathrm{d}v + k \chi_{sr} \Big) \delta\Big(v - \frac{\omega}{k}\Big)$$

(in which $\mathcal{P}$ indicates principal value). Substitution into the first equation (3.2) and integration gives

$$\pi \omega_s^2 F_s' E_1^{\mathrm{H}}(k, kv) + \Big[\int K_s \mathrm{d}v + k \chi_{sr} \Big]_{\omega = kv} |k| E_1(k, kv) = \frac{N e_s}{\mathrm{i}\epsilon_o} g_{s1}.$$

Here kv can be replaced by ω again. $E_1^{\mathrm{H}}(k, \omega)$ is the Hilbert transform of E_1:

$$E_1^\pm = \frac{\pm \mathrm{i}}{2\pi} \int \frac{E_1(k, w)}{\omega_\pm - w} \mathrm{d}w = \frac{1}{2}(E_1 \pm \mathrm{i} E_1^{\mathrm{H}}), \quad E_1^{\mathrm{H}}(k, \omega) = \frac{1}{\pi} \int E_1(k, w) \frac{\mathcal{P}}{\omega - w} \mathrm{d}w.$$

The 'negative' part E_1^- contributes only for negative t, and vanishes since E_1 has only poles in the lower half ω plane. Therefore $E_1^+ = E_1 = \mathrm{i} E_1^{\mathrm{H}}$ and

$$-\mathrm{i}\pi \omega_s^2 F_s'\Big(\frac{\omega}{k}\Big) E_1 + \Big(\int K_s \mathrm{d}v + k \chi_{sr} \Big) |k| E_1 = \frac{N e_s}{\mathrm{i}\epsilon_o} g_{s1}\Big(k, \frac{\omega}{k}\Big).$$

Since $\chi_s^+ = \chi_{sr} - \mathrm{i}\pi(\omega_s^2/k|k|) F_s'(\omega/k)$, this reduces to

$$\int K_s \mathrm{d}v = -k \chi_s^+ + \frac{N e_s}{\mathrm{i}\epsilon_o |k| E_1} g_{s1}\Big(k, \frac{\omega}{k}\Big). \tag{3.5}$$

Summing over s yields

$$k\epsilon^+ E_1 = \frac{Ne}{\mathrm{i}\epsilon_o|k|}\Big[g_{e1} - g_{i1}\Big]_{v=\omega/k}.$$

This is equivalent to (3.4) because the vanishing of E_1^- requires that

$$\int \frac{g_{e1} - g_{i1}}{\omega_- - kv}\mathrm{d}v = 0, \quad \frac{1}{2\pi}\int \frac{g_{e1} - g_{i1}}{\omega_+ - kv}\mathrm{d}v = \frac{1}{\mathrm{i}|k|}\Big[g_{e1} - g_{i1}\Big]_{v=\omega/k}$$

which follows from an expression for E_1^- similar to (3.4) with Ω_-.

The imaginary part of (3.5) for $s = i$ shows that g_{i1} cannot be zero for an electron wave. But $|g_{i1}/E_1|$ can be minimized by setting $\int K_i\mathrm{d}v + k\chi_{ir} = 0$. Then (2.5) is valid for all k and ω,

$$K_i = \omega_i^2 F_i' \frac{\mathcal{P}}{kv - \omega}, \quad K_e = \omega_e^2 F_e' \frac{\mathcal{P}}{kv - \omega} + k\epsilon_r \delta\Big(v - \frac{\omega}{k}\Big) \tag{3.6}$$

and the initial conditions get the form

$$g_{i1} = -\pi\frac{e}{m_i}E_1(k, kv)F_i', \quad g_{e1} = \pi\frac{e}{m_e}E_1(k, kv)F_e' + \frac{\mathrm{i}\epsilon_o}{Ne}|k|kE_1(k, kv)\epsilon_r(k, kv).$$

These initial conditions yield the field (1.5) not just asymptotically but for all t. Other initial conditions that yield the same field for $t > 0$ and a non-vanishing field for $t < 0$ are equally valid for the physics. But, in second-order, $E_1^- \neq 0$ would make the calculation even more complicated.

3.2. Second order

E_0 is found like E_1. The calculation, which differs slightly from the derivation by Best (1985) for a mono-species plasma, is outlined in in Appendix B. The result for the 'positive' part E_0^+, defined as $E_0^+ = (\mathrm{i}/2\pi)\int E_0\mathrm{d}\omega\mathrm{d}\omega'/(\Omega_+ - \omega'')$, is

$$\epsilon^+(k'', \Omega)E_0^+ = \frac{-e}{4\pi m_e}\Big(3 + \frac{k''^2}{k_0^2}\chi_e^+(k'', \Omega)\Big)\frac{k''E_kE_{k'}}{(\Omega - \Omega_1)^2(\Omega - \Omega_2)} + \frac{Ne}{2\pi\epsilon_o k''}\int \frac{g_{e0} - g_{i0}}{\Omega_+ - k''v}\mathrm{d}v, \tag{3.7}$$

with the notation

$$\omega_0 k''/k \approx \omega_0' k''/k' \approx k''u_p - \mathrm{i}0 = \Omega_1, \quad \omega_0 + \omega_0' \approx k''u_g - 2\mathrm{i}\gamma = \Omega_2$$

in which $u_g = \partial_k\omega_r|_{k_0}$ is the group velocity. The poles Ω_1 and Ω_2 are in the lower half of the Ω plane, as causality requires. Again, E_0^+ contributes only for positive t in view of

$$\int \frac{E_0}{\Omega_+ - \omega''}\mathrm{d}\omega\mathrm{d}\omega' = -\mathrm{i}\int_0^\infty \mathrm{d}t\mathrm{e}^{\mathrm{i}\Omega t}\int \mathrm{d}\omega\mathrm{d}\omega' E_0\mathrm{e}^{-\mathrm{i}\omega'' t}.$$

E_0 is not completely determined (Best 1974), but only integrals like (2.7) of the form (G is arbitrary)

$$\int (E_0^+ + E_0^-)G(\Omega)\mathrm{d}\Omega = \int E_0\delta(\Omega - \omega'')G(\Omega)\mathrm{d}\omega\mathrm{d}\omega'\mathrm{d}\Omega = \int E_0 G(\omega'')\mathrm{d}\omega\mathrm{d}\omega' \tag{3.8}$$

will be needed, in which $E_0^- = (-\mathrm{i}/2\pi)\int E_0/(\Omega_- - \omega'')\mathrm{d}\omega\mathrm{d}\omega'$ can be neglected because it contributes only for negative t.

In the calculation of E_0^+, expansions in k'' and Ω are made. Hence only the slow variation of the second-order field is found. The field is switched on at $t = 0$, but the high frequencies in the spectrum of this jump are not included in (3.7).

The solution (3.7) shows secular behaviour associated with the double pole at Ω_1. The secularity can be removed (Best 1999) by a suitable initial disturbance of resonant electrons in second-order, proportional to the derivative of the delta function in velocity space $\delta'(v - u_p)$, to get in (3.7) another secularity which cancels the first one. This procedure is similar to quasilinear theory, which also models the solution to get rid of secular behaviour.

The δ' disturbance causes a dip (Best 1999) in the asymptotic behaviour of the energy density. A $\delta(v - u_p) - g(v)$ disturbance of the electrons is assumed to remove the dip and to adjust the energy density. The normalised smooth g, $\int g\mathrm{d}v = 1$, is subtracted to leave the charge density undisturbed; g represents the part of the electron distribution where the electrons in the δ peak come from.

Ions are not disturbed in second order. Thus the Ansatz for the second-order initial condition amounts to

$$g_{e0} = \frac{\epsilon_o}{Nm_e} E_k E_{k'} [c_0 \delta(v - u_p) - c_0 g + c_1 \delta'(v - u_p)], \quad g_{i0} = 0, \tag{3.9}$$

in which $c_0(k'')$, $c_1(k'')$ and $g(v)$ are to be determined. Equation (3.7) becomes

$$\epsilon^+(k'', \Omega) E_0^+ = \frac{-e}{4\pi m_e} E_k E_{k'} \Big[\Big(3 + \frac{k''^2}{k_0^2} \chi_e^+(k'', \Omega) \Big) \frac{k''}{\Omega - \Omega_2} \frac{1}{(\Omega - \Omega_1)^2}$$

$$- \frac{2c_0}{k''} \Big(\frac{1}{\Omega - \Omega_1} - \int \frac{g \mathrm{d}v}{\Omega_+ - k'' v} \Big) + \frac{2c_1}{(\Omega - \Omega_1)^2} \Big].$$

Since $(k''^2/k_0^2)\chi_e^+(k'', \Omega_1) \approx \chi_e^+(k_0, \omega_{r0}) = -1$, the pole at $\Omega = \Omega_1$ can be made simple by setting $c_1 = -k''/(\Omega_1 - \Omega_2)$. E_0^+ can be written as

$$E_0^+ = \frac{e}{2\pi m_e} \frac{E_k E_{k'}}{\epsilon^+(k'', \Omega)} Y, \quad Y = \frac{k'' D}{(\Omega - \Omega_1)^2} + \frac{c_0}{k''} \Big(\frac{1}{\Omega_- \Omega_1} - \int \frac{g \mathrm{d}v}{\Omega_+ - k'' v} \Big),$$

$$D(\Omega) = \frac{1}{\Omega_1 - \Omega_2} - \frac{3 + (k''^2/k_0^2)\chi_e^+(k'', \Omega)}{2(\Omega - \Omega_2)}, \tag{3.10}$$

with $D(\Omega_1) = 0$.

E_0^+ has poles at $\Omega = \Omega_1$ and $\Omega = \Omega_2$ associated with perturbations moving at the phase and group velocities. The zeros of $\epsilon^+(k'', \Omega)$ are also poles of E_0^+ but these are less interesting for the problem of this paper. There are poles near $\pm\omega_p$, which are neglected together with other fast varying terms in the second-order field. Also, poles near 0 representing damped ion (sound) waves are not related to the electron wave packet. A second-order initial perturbation that does not excite these spurious waves has not been found.

4. Energy and momentum

Substitution of (3.10) in (2.7) using (3.8), and k integration with $k' = k'' - k$ as in (1.6), gives

$$U = \frac{\mathrm{i}m_i}{2\pi m_e} \int_0 U_{k''} \frac{\chi_{ir}(k'', \Omega)}{\epsilon^+(k'', \Omega)} \frac{\Omega^2}{k''} Y \mathrm{e}^{\mathrm{i}(k''x - \Omega t)} \mathrm{d}\Omega \mathrm{d}k'' + \frac{3}{2} U_f,$$

$$P = \frac{\mathrm{i}m_i}{\pi m_e} \int_0 U_{k''} \frac{\chi_{ir}(k'', \Omega)}{\epsilon^+(k'', \Omega)} \Omega Y \mathrm{e}^{\mathrm{i}(k''x - \Omega t)} \mathrm{d}\Omega \mathrm{d}k''.$$

The 0 attached to the integrals indicates that the slowly varying part is considered defined by $|\Omega| << \omega_p$.

If $\Omega^2/k''^2 << \theta/m_e$, then Y does not depend on Ω, and the fraction χ_{ir}/ϵ^+ is proportional to $1/\Omega^2$, reaching a maximum of order unity for $\Omega^2/k''^2 = \mathcal{O}(\theta/m_i)$. U has in the integral an additional factor Ω^2, so the small part of the Ω integration region in which Ω/k'' is in the range of ion velocities can be neglected, which justifies the approximation $\chi_{ir}(k'', \Omega) = -\omega_i^2/\Omega^2$. This does not apply to P. One has

$$U = \frac{\omega_p^2}{2\pi\mathrm{i}} \int_0 \frac{U_{k''}}{\epsilon^+(k'', \Omega)} \frac{Y}{k''} \mathrm{e}^{\mathrm{i}(k''x - \Omega t)} \mathrm{d}\Omega \mathrm{d}k'' + \frac{3}{2} U_f,$$

$$P = \frac{\omega_p^2}{\pi\mathrm{i}} \int_0 \frac{U_{k''}}{\epsilon^+(k'', \Omega)} \frac{Y}{\Omega} \Big[\frac{-\Omega^2}{\omega_i^2} \chi_{ir}(k'', \Omega) \Big] \mathrm{e}^{\mathrm{i}(k''x - \Omega t)} \mathrm{d}\Omega \mathrm{d}k''. \tag{4.1}$$

The expression between square brackets in (4.1) is equal to unity everywhere except in the small 'ionic' region $\Omega^2/k''^2 = \mathcal{O}(\theta/m_i)$. Near zero it is proportional to Ω^2, so there is no pole at $\Omega = 0$ in P. Equations (4.1) first are studied analytically, and then a numerical integration is presented.

The pole at Ω_1 in Y determines the asymptotic behaviour of P and U when $\gamma t >> 1$. To calculate the residue, the relation

$$D'(\Omega_1) = \frac{-2\mathrm{i}\gamma\nu}{\Omega_1(\Omega_1 - \Omega_2)^2}, \quad \nu = \frac{u_p}{u_p - u_g} \tag{4.2}$$

is used (see Appendix C). The contribution of this pole is

$$P_1 = \frac{2}{u_p} U_1, \quad U_1 = \int U_{k''} A_1 \mathrm{e}^{\mathrm{i}k''(x - u_p t)} \mathrm{d}k'', \quad A_1 = \frac{\omega_p^2}{\omega_{r0}^2} \Omega_1^2 D'(\Omega_1) + \frac{\omega_p^2}{k_0^2} c_0. \tag{4.3}$$

A_1 is rewritten as

$$A_1 = r\nu^2 \frac{-2\mathrm{i}hk''}{(k''+2\mathrm{i}h)^2} + ru_p^2 c_0, \quad r = \frac{\omega_p^2}{\omega_{r0}^2}, \quad h = \frac{\gamma}{u_p - u_g}$$

The pole at $k'' = -2\mathrm{i}h$ in the lower half complex k'' plane contributes only if $x < u_p t$. Integration of the first term of A_1 can be done for a short wave packet ($<< h^{-1}$), i.e. for small Landau damping.

$$U_1 = -4\pi r\nu^2 h\big[\mathrm{d}_{k''} k'' \mathrm{e}^{\mathrm{i}k''(x-u_p t)} U_{k''}\big]_{k''=-2\mathrm{i}h}$$

$$\approx -2r\nu^2 h\big[1 + 2h(x-u_p t)\big]\mathrm{e}^{2h(x-u_p t)} \int U_0 \mathrm{d}x \quad \text{if } x < u_p t,\ c_0 = 0,$$

using $2\pi U_{k''=-2\mathrm{i}h} = \int U_0 \mathrm{e}^{-2hx}\mathrm{d}x \approx \int U_0 \mathrm{d}x$. Thus the asymptotic form of P and U, if $c_0 = 0$, is like $-(1+x)\mathrm{e}^x$ for $x < 0$, with a dip between -1 and 0. There is no secular behaviour. The dip can be removed by a suitable choice of c_0 as will be shown in the next section.

The total (asymptotic) energy and momentum are (using $\int U_0 \mathrm{d}x = 2\pi U_{k''=0}$)

$$\int U \mathrm{d}x = 2\pi\big[U_{k''} A_1\big]_{k''=0} = ru_p^2 c_0(0) \int U_0 \mathrm{d}x, \quad \int P \mathrm{d}x = \frac{2}{u_p} \int U \mathrm{d}x. \tag{4.4}$$

The energy and momentum densities of the initial perturbation (3.9) are

$$U(x,0) = \int \frac{m_e}{2} v^2 N g_{e0} \mathrm{e}^{\mathrm{i}k''x} \mathrm{d}v\mathrm{d}k\mathrm{d}k' = \int U_{k''}(c_0 u_p^2 - c_0 \textstyle\int v^2 g \mathrm{d}v - 2c_1 u_p)\mathrm{e}^{\mathrm{i}k''x}\mathrm{d}k'',$$

$$P(x,0) = \int m_e v N g_{e0} \mathrm{e}^{\mathrm{i}k''x} \mathrm{d}v\mathrm{d}k\mathrm{d}k' = 2\int U_{k''}(c_0 u_p - c_0 \textstyle\int v g \mathrm{d}v - c_1)\mathrm{e}^{\mathrm{i}k''x}\mathrm{d}k''.$$

The total (initial) energy and momentum ($k'' = 0$) amount to

$$\int U \mathrm{d}x = c_0(0)(u_p^2 - \textstyle\int v^2 g \mathrm{d}v) \int U_0 \mathrm{d}x, \quad \int P \mathrm{d}x = 2c_0(0)(u_p - \textstyle\int v g \mathrm{d}v) \int U_0 \mathrm{d}x.$$

Comparison with (4.4) sets the moments $\int v g \mathrm{d}v = u_p(1-r)$, $\int v^2 g \mathrm{d}v = u_p^2(1-r)$. Hence

$$U(x,0) = \int U_{k''}(ru_p^2 c_0 - 2u_p c_1)\mathrm{e}^{\mathrm{i}k''x}\mathrm{d}k'', \quad P(x,0) = 2\int U_{k''}(ru_p c_0 - c_1)\mathrm{e}^{\mathrm{i}k''x}\mathrm{d}k''. \tag{4.5}$$

For the numerical calculation g is chosen to be Maxwellian, with the above moments:

$$g = (2\pi u_p u_0)^{-1/2} \mathrm{e}^{-(v-u_0)^2/2u_p u_0}, \quad u_0 = u_p(1-r). \tag{4.6}$$

Y in (3.10) can now be written as (see Appendix C)

$$Y = \frac{k''D}{(\Omega - \Omega_1)^2} + \frac{c_0}{k''}\frac{1}{\Omega_- \Omega_1} - \frac{c_0}{k''}\frac{1}{\Omega_- \Omega_0}\Big(1 - \frac{k''^2 u_p u_0}{\omega_p^2}\chi_0^+(k'', \Omega - \Omega_0)\Big),$$

$$\Omega_0 = k''u_0 - \mathrm{i}0, \quad \chi_0^+(k,\omega) = \frac{\omega_p^2}{k\sqrt{2\pi u_p u_0}} \int \frac{\partial_v \mathrm{e}^{-v^2/2u_p u_0}}{\omega_+ - kv}\mathrm{d}v. \tag{4.7}$$

$\Omega = \Omega_0$ is not a pole because $\chi_0^+(k'', 0) = \omega_p^2/k''^2 u_p u_0$.

The pole at Ω_2 in D in (3.10) is not useful for contour integration. Its residue is zero in the limit $k'' \to 0$ as $\chi_e^+(k'', \Omega_2)$ blows up like $\exp(\gamma^2/k''^2)$. The limit $\epsilon^+(k'', \Omega) \to 1 - \omega_p^2/\Omega^2$ is not valid if $-3\pi/4 < \arg\Omega < -\pi/4$.

4.1. Numerical integration

The contributions (4.3) of the pole at Ω_1 are taken apart in (4.1) to eliminate singularities in the Ω integration:

$$U = U_1 + \frac{\mathrm{i}}{2\pi}\int_0 \Big(\frac{A-A_1}{\Omega-\Omega_1} + \frac{B}{\Omega-\Omega_0}\Big)U_{k''}\mathrm{e}^{\mathrm{i}(k''x-\Omega t)}\mathrm{d}\Omega\mathrm{d}k'' + \frac{3}{2}U_f,$$

$$P = P_1 + \frac{\mathrm{i}}{\pi u_p}\int_0 \Big(\frac{AC-A_1}{\Omega-\Omega_1} + \frac{BC}{\Omega-\Omega_0}\Big)U_{k''}\mathrm{e}^{\mathrm{i}(k''x-\Omega t)}\mathrm{d}\Omega\mathrm{d}k'',$$

$$A(\Omega) = \frac{-\omega_p^2}{\epsilon^+(k'',\Omega)}\Big(\frac{D}{\Omega-\Omega_1} + \frac{c_0}{k''^2}\Big), \quad A_1 = A(\Omega_1) = r\Omega_1^2 D'(\Omega_1) + ru_p^2 c_0,$$

$$B(\Omega) = \frac{c_0}{\epsilon^+(k'',\Omega)}\Big(\frac{\omega_p^2}{k''^2} - u_p u_0 \chi_0^+(k'',\Omega-\Omega_0)\Big), \quad C(\Omega) = -\frac{\Omega\Omega_1}{\omega_i^2}\chi_{ir}(k'',\Omega), \tag{4.8}$$

in which $B(\Omega_0) = 0$ and $C(\Omega) = \Omega_1/\Omega$ except for small $\Omega^2 = \mathcal{O}(k''^2\theta/m_i)$. A, A_1, B and C are dimensionless. More dimensionless variables are introduced:

$$\kappa = \frac{k''}{\alpha}, \quad \delta = \frac{\gamma}{\alpha(u_p-u_g)}, \quad \tau = \alpha(u_p-u_g)t, \quad \beta = \frac{\Omega}{\alpha(u_p-u_g)},$$

in which α is a measure for the width of the peaks of the k-spectrum E_k, so the length of the packet is of order α^{-1}. The susceptibilities χ are expressed in terms of the plasma dispersion function w (Abramowitz and Stegun 1972, see also Appendix C):

$$\chi_s^+(k'',\Omega) = \frac{k_0^2}{k''^2}p(1+\mathrm{i}z_s w_s), \quad p = \frac{Ne^2}{\epsilon_o\theta k_0^2}, \quad w_s = w(z_s),$$

$$w(z) = \frac{\mathrm{i}}{\sqrt{\pi}}\int\frac{\mathrm{e}^{-t^2}\mathrm{d}t}{z_+ - t}, \quad z_s = \frac{\Omega}{k''}\sqrt{\frac{m_s}{2\theta}}, \quad z_e = \frac{\beta}{\nu\kappa}\sqrt{\frac{p}{2r}},$$

$$\chi_0^+(k'',\Omega-\Omega_0) = \frac{k_0^2}{k''^2}p_0(1+\mathrm{i}z_0 w_0), \quad p_0 = \frac{Ne^2}{\epsilon_o m_e u_p u_0 k_0^2} = \frac{r}{1-r},$$

$$w_0 = w(z_0), \quad z_0 = \frac{\Omega-\Omega_0}{k''\sqrt{2u_p u_0}} = \frac{\beta-\beta_0}{\nu\kappa\sqrt{2-2r}}$$

in which $\beta_0 = \nu\kappa(1-r)$ and $z_+ = z + \mathrm{i}0$ in w applies to positive κ, sufficient for the κ integration below in view of symmetry (real Fourier transform). In terms of these variables, (4.5) and (4.8) read

$$U(x,0) = \int\Big(ru_p^2c_0 + \frac{2\nu\kappa}{\kappa+2\mathrm{i}\delta}\Big)\alpha U_{k''}e^{\mathrm{i}\kappa\alpha x}\mathrm{d}\kappa, \quad P(x,0) = \frac{2}{u_p}\int\Big(ru_p^2c_0 + \frac{\nu\kappa}{\kappa+2\mathrm{i}\delta}\Big)\alpha U_{k''}e^{\mathrm{i}\kappa\alpha x}\mathrm{d}\kappa,$$

$$U = \int\Big[A_1 + \frac{\mathrm{i}}{2\pi}\int\Big(\frac{A-A_1}{\beta-\beta_1} + \frac{B}{\beta-\beta_0}\Big)\mathrm{e}^{-\mathrm{i}\beta\tau}\mathrm{d}\beta\Big]\alpha U_{k''}\mathrm{e}^{\mathrm{i}\kappa\alpha x}\mathrm{d}\kappa + \frac{3}{2}U_f,$$

$$P = \frac{2}{u_p}\int\Big[A_1 + \frac{\mathrm{i}}{2\pi}\int\Big(\frac{AC-A_1}{\beta-\beta_1} + \frac{BC}{\beta-\beta_0}\Big)\mathrm{e}^{-\mathrm{i}\beta\tau}\mathrm{d}\beta\Big]\alpha U_{k''}\mathrm{e}^{\mathrm{i}\kappa\alpha x}\mathrm{d}\kappa,$$

$$A(\beta) = \frac{-r}{p\sum(1+\mathrm{i}z_s w_s)}\Big(\frac{\nu^2\kappa^2\Delta}{\beta-\beta_1} + u_p^2c_0\Big), \quad A_1 = A(\beta_1) = ru_p^2c_0 - r\nu^2\frac{2\mathrm{i}\delta\kappa}{(\kappa+2\mathrm{i}\delta)^2},$$

$$B(\beta) = \frac{ru_p^2c_0z_0w_0}{\mathrm{i}p\sum(1+\mathrm{i}z_s w_s)}, \quad C(\beta) = \frac{-\beta}{r\nu\kappa}\frac{m_i}{m_e}p(1+\mathrm{i}z_i w_i), \quad \Delta(\beta) = \frac{1}{\kappa+2\mathrm{i}\delta} - \frac{3+p+\mathrm{i}pz_e w_e}{2(\beta-\beta_2)} \tag{4.9}$$

in which $\beta_1 = \nu\kappa - \mathrm{i}0$ and $\beta_2 = (\nu-1)\kappa - 2\mathrm{i}\delta$. B contains a factor $\beta-\beta_0$ in z_0. $1+\mathrm{i}zw \to -1/2z^2$ for large real z, hence $C = \nu\kappa/\beta$ except in the region $\beta^2/\kappa^2 = \mathcal{O}(m_e/pm_i)$, which, in the computer code, is the point $\beta = 0$; $C(0) = 0$. In $\sum(1+\mathrm{i}z_s w_s)$, the ion contribution is negligible except at $\beta = 0$; $\chi_i^+(k'',0) = \chi_e^+(k'',0)$.

For the κ integration, a Gaussian k spectrum is taken,

$$E_k = \frac{a}{4\alpha\sqrt{\pi}}\Big(\mathrm{e}^{-(k-k_0)^2/4\alpha^2} + \mathrm{e}^{-(k+k_0)^2/4\alpha^2}\Big),$$

corresponding to a Gaussian wave packet

$$E = a\mathrm{e}^{-\alpha^2(x-u_g t)^2-\gamma t}\cos(k_0 x - \omega_{r0} t) \tag{4.10}$$

with field energy density

$$U_f = U_0(x-u_g t)\mathrm{e}^{-2\gamma t}, \quad U_0(x) = \frac{\epsilon_o}{4}a^2\mathrm{e}^{-2\alpha^2 x^2}, \quad U_{k''} = \frac{\epsilon_o a^2}{8\alpha\sqrt{2\pi}}\mathrm{e}^{-\kappa^2/8}.$$

Integrations in (4.9) are done by Fast Fourier Transform (Press et al. 1987), using $\delta = 0.2$. Numerical solution of the dispersion equation for $p = 18$ yields $r = 0.838$, $\nu = 1.239$ and $\gamma/\omega_p = 0.0010$. Appendix D contains additional computational remarks.

The figures show the development of U and $Pu_p/2$, divided by the maximum value of the field energy density $U_0(0) = \epsilon_o a^2/4$, as a function of αx for several values of γt. Figure 1 shows U for $c_0 = 0$, i.e. zero total energy. A dip is seen to develop in the precursor to the right in figure 1. P also develops such a dip (not shown).

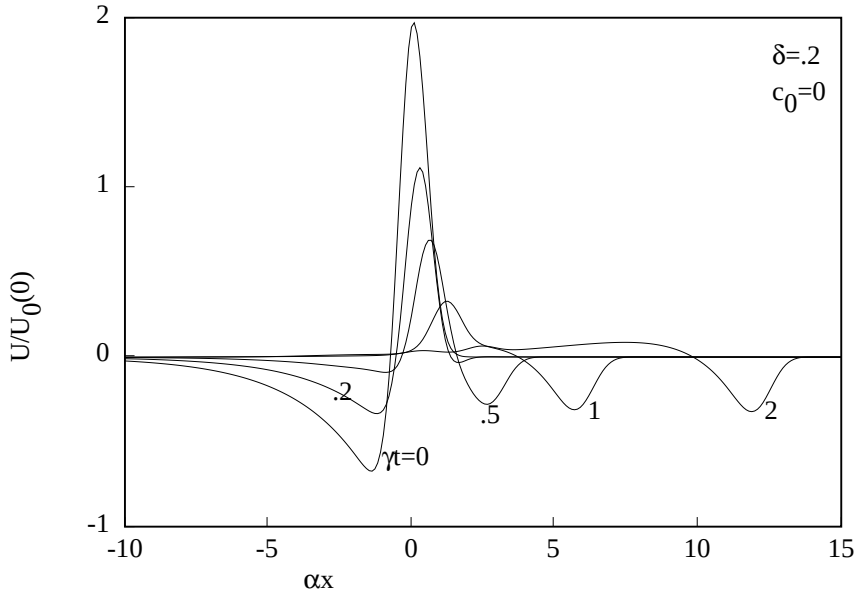

Figure 1: Energy density U of a Gaussian wave packet (4.10) if $c_0 = 0$ in the initial value (3.9). $U_0(0)$ is the peak value of the field energy density.

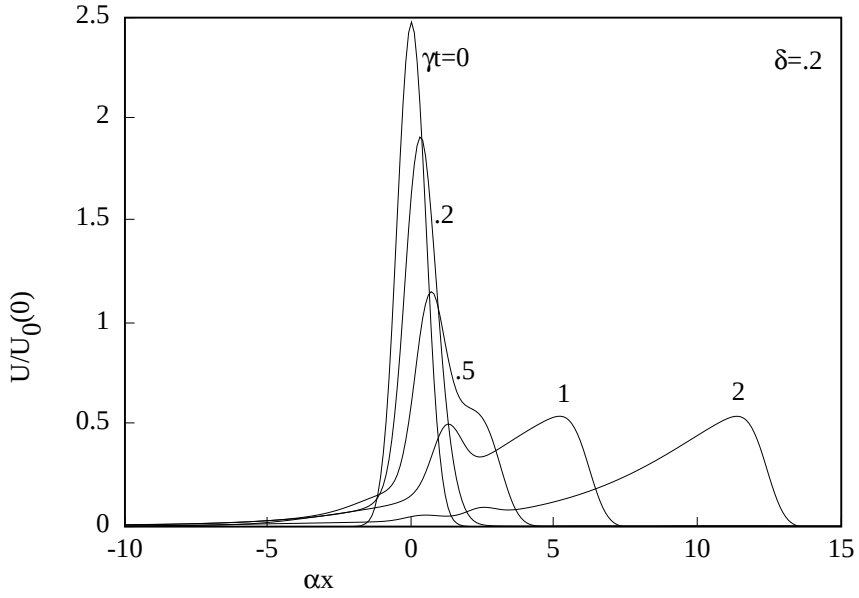

Figure 2: As figure 1, but for c_0 given by (4.11). Wave energy is carried away by resonant electrons.

By modifying c_0,

$$ru_p^2 c_0 = 2\nu \frac{2\mathrm{i}\delta}{\kappa + 2\mathrm{i}\delta}, \tag{4.11}$$

the natural evolution shown in figure 2 can be obtained for U. There is no longer any dip in the precursor. Also, the negative tail in the initial profile (to the left in figure 1) is removed, which means that initially the disturbance is confined to its proper region.

Figure 3 shows P for the same c_0. It has some strange behaviour near $x = 0$, due to lowest-order approximations for the ion component of P. Terms of order m_e/m_i have been neglected throughout. As the analysis is not accurate in the ionic region, which is not important for the electronic wave anyway, it appears reasonable to leave out the ionic region from the integration.

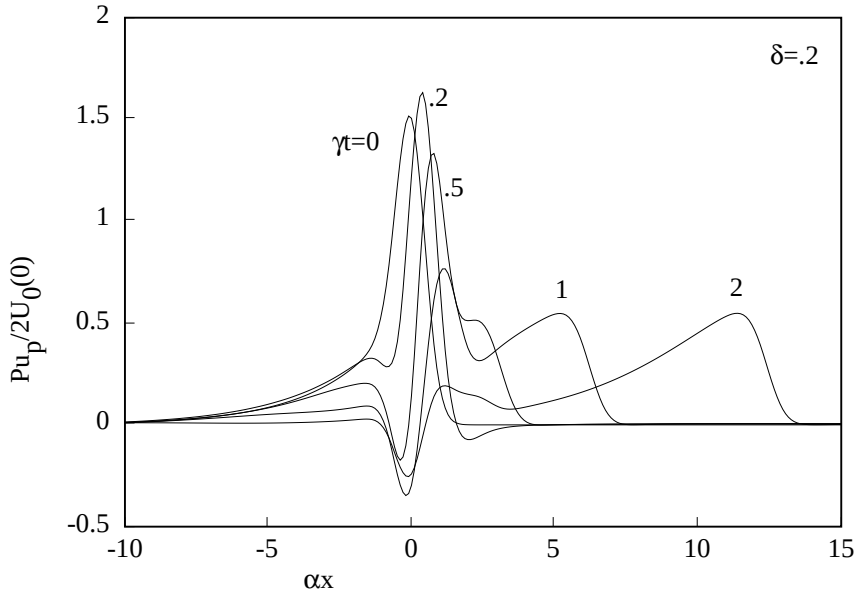

Figure 3: Wave momentum P of the same packet and c_0 as in figure 2. P is multiplied by $u_p/2$, half the phase velocity.

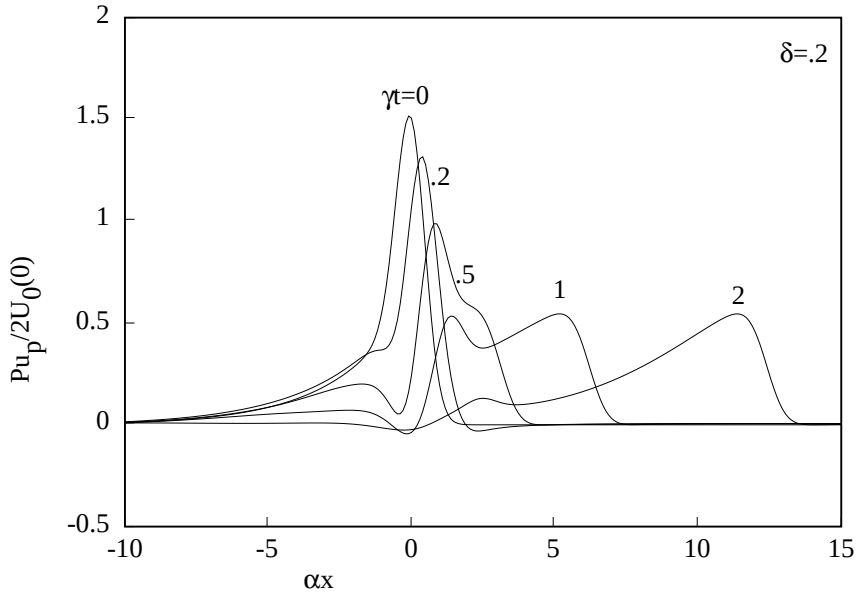

Figure 4: As figure 3, except for the low-frequency cut-off (4.12).

Figure 4 shows the result, obtained by replacing $C = \beta_1/\beta$ by

$$C = \frac{\beta/\beta_1}{0.02 + \beta^2/\beta_1^2} \tag{4.12}$$

corresponding to cut-off at about 1/7 of the phase velocity of the wave.

The total energy and momentum are

$$\int U\mathrm{d}x = 2\nu \int U_0 \mathrm{d}x, \quad \int P\mathrm{d}x = \frac{2}{u_p}\int U\mathrm{d}x. \tag{4.13}$$

The total energy is in agreement with fluid theory as $\omega\partial_\omega\epsilon_r = 2\nu$ at resonance (see Appendix C), but the total momentum is twice as large.

5. Conclusions

The Vlasov-Poisson system (1.1) can be solved under arbitrary initial conditions at first and second orders in E. At first order, initial conditions are chosen to excite an electrostatic wave packet (1.5) with minimal ion disturbance. Second-order expressions are derived for the slowly varying parts of the field (3.7) and the energy and momentum densities (4.1).

To remove secular behaviour, a suitable initial disturbance of resonant electrons in second order is applied, proportional to the derivative of the delta function in velocity space; see the c_1 term in (3.9). The δ' function can be considered as the zero amplitude limit of flattening of the distribution function in quasilinear theory.

If only the c_1 term is used in the second-order initial condition g_{e0} then the total (i.e. space integrated) wave energy and momentum are zero. Figure 1 shows the evolution of the energy density U in this case.

To fill the dip to the right in figure 1, an extra disturbance is needed which propagates undamped at the phase velocity u_p. This leads in the ansatz (3.9) for g_{e0} to c_0 terms which express a shift of electrons from the bulk of the velocity distribution to the resonant region. Figure 2 shows the evolution of U if c_0 is given by (4.11). The initial wave energy is carried away by resonant electrons, moving at a speed $\alpha u_p/\gamma = \nu/\delta = 6.2$ in dimensionless units used in the figure, while the wave packet damps, propagating at the group velocity $(\nu - 1)/\delta = 1.2$. The proposed form of g_{e0} is not a rigorous result but obtained partly by guesswork and numerical experiment to get a physically appealing evolution.

The resulting wave energy, see (4.13), agrees with dielectric fluid theory. As wave energy and momentum end up in electrons with common speed u_p, their ratio is $u_p/2$, contrary to the factor u_p in fluid theory.

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Appendix A

Lacina (1972) considered a damped travelling wave $E = a\mathrm{e}^{-\gamma t}\sin(kx - \omega t) + E_2$, and calculated U_s and P_s for particle species s with normalized equilibrium velocity distribution F_s. For large t, the result can be written as

$$U_s = U_0\omega_s^2\Big(\int \frac{-vF_s'(v)}{(kv-\omega)^2+\gamma^2}\mathrm{d}v\ (1+\mathrm{e}^{-2\gamma t}) + \frac{1}{\gamma}\frac{2\pi}{|k|}[vF_s'(v)]_{v=\omega/k}\mathrm{e}^{-2\gamma t} + \dots\Big)$$

, and for P_s the same formula holds with $vF_s'(v)$ replaced by $F_s'(v)$. $U_0 = \epsilon_o a^2/4$ is the initial value of the average field energy density $U_f = U_0\mathrm{e}^{-2\gamma t}$, and $\omega_s^2 = N_s e_s^2/\epsilon_o m_s$ is the plasma frequency of particle species s with density N_s, charge e_s and mass m_s. The dots indicate terms of higher order in γ.

In the limit of small $\gamma > 0$ this result can be rewritten by aid of the expansion (in which $\mathcal{P}$ means principal value)

$$\frac{\gamma}{w^2+\gamma^2} = \pi\delta(w) + \gamma\frac{\mathcal{P}}{w^2} - \gamma^2\frac{\pi}{2}\delta''(w) + \dots,$$

which is the Fourier transform (Lighthill 1959) of the expansion of $\mathrm{e}^{-\gamma|u|}$. Then U_s and P_s take the forms

$$U_s = U_0\Big[\partial_\omega\omega\chi_{sr}(1+\mathrm{e}^{-2\gamma t}) + \frac{\omega\chi_{si}}{\gamma}(1-\mathrm{e}^{-2\gamma t})\Big], \quad P_s = U_0\Big[k\partial_\omega\chi_{sr}(1+\mathrm{e}^{-2\gamma t}) + \frac{k\chi_{si}}{\gamma}(1-\mathrm{e}^{-2\gamma t})\Big]$$

in terms of the dielectric susceptibility

$$\chi_s^+ = \chi_{sr} + \mathrm{i}\chi_{si} = \frac{\omega_s^2}{k}\int F_s'\frac{\mathcal{P}}{\omega - kv}\mathrm{d}v - \mathrm{i}\pi\frac{\omega_s^2}{k|k|}F_s'(\frac{\omega}{k}).$$

The wave energy density $U_f + \sum U_s$ and momentum density $\sum P_s$ are constant if (and only if) the dispersion relation $\epsilon_r = 1 + \sum\chi_{sr} = 0$, $\gamma = \sum\chi_{si}/\partial_\omega\epsilon_r$ is satisfied. In the approximation $\epsilon_r = 1 - \sum\omega_s^2/\omega^2 = 1 - \omega_p^2/\omega^2$ the constants are $U_f + \sum U_s = 3U_0$ and $\sum P_s = 4U_0k/\omega_p$. The terms χ_{sr} and χ_{si} can be associated with non-resonant motion and resonant motion respectively. Then the sum of the energy densities of the field and the damped non-resonant motion $\sim \partial_\omega\omega\chi_{sr}\mathrm{e}^{-2\gamma t}$ is $U_f\partial_\omega\omega\epsilon_r$, and the momentum density of the damped non-resonant motion is $U_fk\partial_\omega\epsilon_r$.

Appendix B

To solve for E_0, relations like (3.3) are needed:

$$\int\frac{K_s}{\Omega_+ - k''v}\mathrm{d}v = \frac{k}{k''}\int\frac{K_s}{\Omega_+k/k'' - kv}\mathrm{d}v = \frac{\int K_s\mathrm{d}v + k\chi_s^+(k,\Omega k/k'')}{\Omega_+ - \omega k''/k} = \frac{\int K_s\mathrm{d}v + (k''^2/k)\chi_s^+(k'',\Omega)}{\Omega_+ - \omega k''/k} = A_s \tag{B.1}$$

and

$$\int\frac{L_s}{\Omega_+ - k''v}\mathrm{d}v = \frac{1}{\Omega_+ - \omega''}\int L_s\Big(1 + \frac{k''v - \omega''}{\Omega_+ - k''v}\Big)\mathrm{d}v = \frac{1}{\Omega_+ - \omega''}\int\frac{\partial_v(K_s + K_s')}{\Omega_+ - k''v}\mathrm{d}v$$
$$= \frac{k''}{\Omega_+ - \omega''}\partial_\Omega\int\frac{K_s + K_s'}{\Omega_+ - k''v}\mathrm{d}v = \frac{k''}{\Omega_+ - \omega''}\partial_\Omega(A_s + A_s'), \tag{B.2}$$

with

$$A_s' = \frac{\int K_s'\mathrm{d}v + (k''^2/k')\chi_s^+(k'',\Omega)}{\Omega_+ - \omega'k''/k'}.$$

Division of the second equation (3.2) by $2\pi(\Omega_+ - k''v)$, integration over v using (B.1) and summation over s yields E_0^+ defined as $E_0^+ = (\mathrm{i}/2\pi)\int E_0\mathrm{d}\omega\mathrm{d}\omega'/(\Omega_+ - \omega'')$; thus

$$\frac{\mathrm{i}}{2\pi}\int E_0\frac{K_e'' + K_i''}{\Omega_+ - k''v}\mathrm{d}v\mathrm{d}\omega\mathrm{d}\omega' = k''\epsilon^+(k'',\Omega)E_0^+$$

$$= \frac{e}{4\pi}\int\frac{E_1E_1'}{\Omega_+ - k''v}\Big(\frac{L_e}{m_e} - \frac{L_i}{m_i}\Big)\mathrm{d}v\mathrm{d}\omega\mathrm{d}\omega' + \frac{Ne}{2\pi\epsilon_o}\int\frac{g_{e0} - g_{i0}}{\Omega_+ - k''v}\mathrm{d}v.$$

The E_1E_1' term has to be evaluated. Substitution of (1.5) and (B.2) gives

$$\frac{e}{4\pi}\Big(\frac{\mathrm{i}}{2\pi}\Big)^2\int\frac{E_k}{\omega - \omega_0}\frac{E_{k'}}{\omega' - \omega_0'}\frac{k''}{\Omega_+ - \omega''}\partial_\Omega\Big(\frac{A_e + A_e'}{m_e} - \frac{A_i + A_i'}{m_i}\Big)\mathrm{d}\omega\mathrm{d}\omega'.$$

Take first the case (Best 1974) where $k/k'' > 1$, which implies that $k'k'' < 0$, and integrate around the lower half ω plane, which contains only the pole ω_0, and then around the upper half ω' plane, which contains only the pole $\Omega - \omega_0$:

$$\frac{e}{4\pi}\frac{k''E_kE_{k'}}{\Omega - \omega_0 - \omega_0'}\Big[\partial_\Omega\Big(\frac{A_e + A_e'}{m_e} - \frac{A_i + A_i'}{m_i}\Big)\Big]_{\omega=\omega_0,\omega'=\Omega-\omega_0}.$$

Terms with $\partial_\Omega\chi_s^+$ cancel since $(\Omega k - \omega k'')^{-1} + (\Omega k' - \omega' k'')^{-1} = 0$ for $\omega = \omega_0$, $\omega' = \Omega - \omega_0$. Hence

$$\frac{-e}{4\pi}\frac{k''E_kE_{k'}}{\Omega - \omega_0 - \omega_0'}\frac{1}{(\Omega k - \omega_0 k'')^2}\Big(\frac{X_e}{m_e} - \frac{X_i}{m_i}\Big),$$

in which $X_s = k^2\int K_s(k,v,\omega_0)\mathrm{d}v + k'^2\int K_s(k',v,\Omega-\omega_0)\mathrm{d}v + k''^3\chi_s^+(k'',\Omega)$. Using (2.5) with $\chi_{ir} = -\omega_i^2/\omega^2$ for the integrals in X_i, writing $k' = -k + k''$, and expanding in k'' and Ω, some algebra yields that $X_e \approx 3k''k^2 + k''^3\chi_e^+(k'',\Omega)$ and X_im_e/m_i is negligible. This leads to (3.7) which is symmetric in dashed and undashed quantities and valid also for the case $k'/k'' > 1$.

Appendix C

To derive partial derivatives, χ_s^+ is written as (Abramowitz and Stegun 1972)

$$\chi_s^+(k,\omega) = p(1+\mathrm{i}z_s w), \quad w = \frac{\mathrm{i}}{\sqrt{\pi}}\int \frac{\mathrm{e}^{-t^2}\mathrm{d}t}{z_{s+}-t}, \quad p = \frac{Ne^2}{\epsilon_o\theta k^2}, \quad z_s = \frac{\omega}{k}\sqrt{\frac{m_s}{2\theta}}$$

if $k>0$. For $w(z)$, we have $w' = 2(\mathrm{i}-zw)$. Hence

$$\omega\partial_\omega\chi_s^+ = \mathrm{i}\omega p(w+z_s w')\partial_\omega z_s = (1-2z_s^2)\chi_s^+ - p,$$

$$k\partial_k\chi_s^+ = k(1+\mathrm{i}z_s w)\partial_k p + \mathrm{i}kp(w+z_s w')\partial_k z = (2z_s^2-3)\chi_s^+ + p.$$

Taking the real part of these equations and $\omega=\omega_r$, a relation between ν, p and r can be derived using $\sum\chi_{sr}=-1$, $\chi_{ir}=-\omega_i^2/\omega_r^2$, $2z_i^2\chi_{ir}=-p$ and $2z_e^2\chi_{er}=-p/r+\mathcal{O}(m_e/m_i)$. Thus

$$\frac{u_g}{u_p} = -\frac{k\partial_k\epsilon_r}{\omega\partial_\omega\epsilon_r}\Big|_{\omega_r} = \frac{p-r(p+3)}{p-r(p+1)}$$

which implies $2\nu = (p/r)-p-1 = \partial_\omega\omega\epsilon_r\big|_{\omega_r} = \omega_r\partial_\omega\epsilon_r\big|_{\omega_r}$. For large p, $r\approx 1-3/p$.

The derivative of D at Ω_1 is

$$\partial_\Omega D(\Omega)\Big|_{\Omega_1} = \frac{1}{(\Omega_1-\Omega_2)^2} - \frac{k''^2}{2k_0^2}\partial_\Omega\chi_e^+(k'',\Omega)\Big|_{\Omega_1}\frac{1}{\Omega_1-\Omega_2}$$

$$= \frac{1}{(\Omega_1-\Omega_2)^2} - \frac{k''^2}{2k_0^2}\frac{1}{\Omega_1}\left[\Big(1-\frac{\Omega_1^2 m_e}{k''^2\theta}\Big)\frac{-k_0^2}{k''^2} - \frac{\omega_e^2 m_e}{k''^2\theta}\right]\frac{1}{\Omega_1-\Omega_2}$$

$$= \frac{1}{(\Omega_1-\Omega_2)^2} + \frac{1+p-p/r}{2\Omega_1(\Omega_1-\Omega_2)} = \frac{1}{(\Omega_1-\Omega_2)^2} + \frac{-\nu}{\Omega_1(\Omega_1-\Omega_2)}$$

which equals (4.2).

To evaluate

$$\int\frac{g\mathrm{d}v}{\Omega_+ - k''v} = \frac{1}{\sqrt{2\pi u_p u_0}}\int\frac{\mathrm{e}^{-v^2/2u_p u_0}}{\Omega_+ - k''u_0 - k''v}\mathrm{d}v$$

use

$$\int\frac{F_e'\mathrm{d}v}{\omega_+ - kv} = \frac{-m_e}{k\theta}\int\frac{kv-\omega+\omega}{\omega_+-kv}F\mathrm{d}v, \quad \omega\int\frac{F_e\mathrm{d}v}{\omega_+-kv} = 1-\frac{k^2\theta}{\omega_p^2 m_e}\chi_e^+.$$

Similarly,

$$(\Omega-\Omega_0)\int\frac{g\mathrm{d}v}{\Omega_+-k''v} = 1-\frac{k''^2u_p u_0}{\omega_p^2}\chi_0^+(k'',\Omega-\Omega_0),$$

which leads to (4.7).

Appendix D

In numerical calculations, $w(z)=\sqrt{\pi}\mathrm{e}^{-z^2}\mathrm{erfc}(-\mathrm{i}z)$ is evaluated using the expansion (7.1.29) of Abramowitz and Stegun (1972) and the approximation

$$w = \mathrm{i}z\Big(\frac{a}{z^2-0.25/a}+\frac{1-a}{z^2-3+0.25/a}\Big), \quad a=0.90824825$$

for large z; cf. the equations below Table 7.9 and note the difference by a factor $\sqrt{\pi}$ in the definition used in this paper.

In the β integrals at a few points near β_1, numerical accuracy is insufficient because in Δ, and again in $A-A_1$ and $AC-A_1$, the difference of nearly equal numbers is involved. At these points, interpolation is applied.

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